ISRAEL JOURNAL OF MATHEMATICS, Vol. 62, No. 3, 1988

THE LINEAR ARBORICITY OF GRAPHS

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ABSTRACT

A *linear forest* is a forest in which each connected component is a path. The *linear arboricity* $\mathrm{la}(G)$ of a graph G is the minimum number of linear forests whose union is the set of all edges of G. The *linear arboricity conjecture* asserts that for every simple graph G with maximum degree $\Delta = \Delta(G)$,

$$\mathrm{la}(G) \leqq \left\lceil \frac{\Delta + 1}{2} \right\rceil .$$

Although this conjecture received a considerable amount of attention, it has been proved only for $\Delta \leqq 6$, $\Delta = 8$ and $\Delta = 10$, and the best known general upper bound for $\mathrm{la}(G)$ is $\mathrm{la}(G) \leqq \lceil 3\Delta/5 \rceil$ for even Δ and $\mathrm{la}(G) \leqq \lceil (3\Delta + 2)/5 \rceil$ for odd Δ. Here we prove that for every $\varepsilon > 0$ there is a $\Delta_0 = \Delta_0(\varepsilon)$ so that $\mathrm{la}(G) \leqq (\frac{1}{2} + \varepsilon)\Delta$ for every G with maximum degree $\Delta \geqq \Delta_0$. To do this, we first prove the conjecture for every G with an even maximum degree Δ and with *girth* $g \geqq 50\Delta$.

1. Introduction

All graphs considered here are finite, undirected and simple, i.e., have no loops and no multiple edges, unless otherwise specified. A *linear forest* is a forest in which every connected component is a path. The *linear arboricity* $\mathrm{la}(G)$ of a graph G is the minimum number of linear forests in G, whose union is the set of all edges of G. This notion was introduced by Harary in [H] as one of the covering invariants of graphs. The following conjecture, known as the *linear arboricity conjecture*, was raised in [AEH1]:

[†] Research supported in part by Allon Fellowship, by a Bat Sheva de Rothschild grant, by the Fund for Basic Research administered by the Israel Academy of Sciences and by a B.S.F. Bergmann Memorial grant.

Received January 19, 1988

CONJECTURE 1.1 (The linear arboricity conjecture). The linear arboricity of every d-regular graph is $\lceil (d+1)/2 \rceil$.

Notice that since every d-regular graph G on n vertices has $nd/2$ edges, and every linear forest in it has at most $n-1$ edges, the inequality

$$\mathrm{la}(G) \geqq \frac{nd}{2(n-1)} > \frac{d}{2}$$

is immediate. Since $\mathrm{la}(G)$ is an integer this gives $\mathrm{la}(G) \geqq \lceil (d+1)/2 \rceil$. The difficulty in Conjecture 1.1 lies in proving the converse inequality: $\mathrm{la}(G) \leqq \lceil (d+1)/2 \rceil$. Note also that since every graph G with maximum degree Δ is a subgraph of a Δ-regular graph (which may have more vertices, as well as more edges than G), the linear arboricity conjecture is equivalent to the statement that the linear arboricity of every graph G with maximum degree Δ is at most $\lceil (\Delta+1)/2 \rceil$.

Although the linear arboricity conjecture received a considerable amount of attention, it has been proved only in a few special cases. The conjecture was proved for $d=3, 4$ by Akiyama, Exoo and Harary in [AEH1], [AEH2] (see also [AC] for a short proof). The cases $d=5, 6$ were solved independently by Enomoto [E], Peroche [P] and Tomasta [T] (only for $d=6$). The case $d=8$ was proved by Enomoto and Peroche [EP], and the case $d=10$ was proved by Guldan [G1]. In the general case, as mentioned above, the linear arboricity of every d-regular graph is trivially at least $\lceil (d+1)/2 \rceil$. In [AEH2] it was shown that for each such G, $\mathrm{la}(G) \leqq \lceil 3\lceil d/2 \rceil /2 \rceil$. This was improved in [P] to $\mathrm{la}(G) \leqq \lceil 2d/3 \rceil$ for even d and $\mathrm{la}(G) \leqq \lceil (2d+1)/3 \rceil$ for odd d. A further improvement is given in [EP], where it is shown that $\mathrm{la}(G) \leqq \lceil 5d/8 \rceil$ for even d and $\mathrm{la}(G) \leqq \lceil (5d+3)/8 \rceil$ for odd d. Presently, the best known general bound, proved in [G2], is $\mathrm{la}(G) \leqq \lceil 3d/5 \rceil$ for even d and $\mathrm{la}(G) \leqq \lceil (3d+2)/5 \rceil$ for odd d.

In this paper we prove that for every $\varepsilon > 0$ there is a $d_0 = d_0(\varepsilon)$ such that for every $d \geqq d_0$ the linear arboricity of every d-regular graph is smaller than $(\frac{1}{2}+\varepsilon)d$. To establish this, we first prove that the linear arboricity conjecture holds for every graph with an even degree of regularity d and with girth $g \geqq 50d$. Similarly, we establish the conjecture for every graph with an odd degree of regularity d and with girth $g \geqq 100d$ that contains a perfect matching.

Our method differs considerably from the ones used in the previous works on the problem, and relies heavily on probabilistic arguments.

The paper is organized as follows. In Section 2 we prove Conjecture 1.1 for graphs with an even degree of regularity and sufficiently large girth. In

Section 3 we show that for every $\varepsilon > 0$ and for every d-regular graph G, $\frac{1}{2}d < \mathrm{la}(G) < (\frac{1}{2} + \varepsilon)d$ provided $d > d_0(\varepsilon)$. In the final Section 4 we describe briefly various related results that can be proved using our method.

2. Graphs with large girth

In this section we show that Conjecture 1.1 holds for all graphs with an even degree of regularity and with sufficiently large girth. Specifically, we prove the following result.

THEOREM 2.1. *Let G be a d-regular graph, where d is an even integer, with girth $g \geqq 50d$. Then*

$$\mathrm{la}(G) = \frac{d}{2} + 1.$$

Moreover, the edges of G can be covered by $d/2$ linear forests and one matching.

REMARK 2.2. The constant 50 can be somewhat reduced; we make no attempt in optimizing the constants here and in the following results.

To prove Theorem 2.1 we need the following result, known as the Lovász Local Lemma, first proved in [EL]. We urge the readers who are unfamiliar with the extemely simple proof of this useful result to consult [EL] or [S].

LEMMA 2.3 (Lovász Local Lemma). *Let $A_1, A_2, \dots, A_n$ be events in a probability space. A graph $T = (V(T), E(T))$ on the set of vertices $V(T) = \{1, 2, \dots, n\}$ is called a* dependency graph *for $A_1, \dots, A_n$ if for all i, A_i is mutually independentof all A_j with $\{i, j\} \notin E(T)$. Assume there exist n numbers $x_1, x_2, \dots, x_n \in [0, 1)$ such that*

$$\Pr(A_i) < x_i \prod_{\{i,j\} \in E(T)} (1 - x_j)$$

for all i, $1 \leqq i \leqq n$. Then

$$\Pr\left(\bigwedge_{i=1}^{n} \bar{A}_i \right) > \prod_{i=1}^{n} (1 - x_i).$$

In particular, with positive probability no A_i occurs. □

Using the last lemma, we now prove the following Proposition, which is the main tool in the proof of Theorem 2.1.

PROPOSITION 2.4. *Let $H = (V, E)$ be a graph with maximum degree d, and* *let $V = V_1 \cup V_2 \cup \cdots \cup V_r$ be a partition of V into r pairwise disjoint sets. Suppose each set V_i is of cardinality $|V_i| \geqq 25d$. Then there is an independent set of vertices $W \subseteq V$, that contains at least one vertex from each V_i.*

REMARK 2.5. The constant 25 can be somewhat reduced. As mentioned above, we make no attempt in optimizing the constants. It is, however, easy to find some simple examples showing that it cannot be reduced to $\frac{3}{2}$ (or less). A version of Proposition 2.4 for hypergraphs can be formulated and proved, by an easy modification of the proof below. For our purposes here, the present version suffices.

PROOF OF PROPOSITION 2.4. Clearly we may assume that each set V_i is of cardinality precisely $g = 25d$ (otherwise, simply replace each V_i by a subset of cardinality g of it, and replace H by its induced subgraph on the union of these r new sets). Put $p = 1/25d$, and let us pick each vertex of H, randomly and independently, with probability p. Let W be the random set of all vertices picked. To complete the proof we show that with positive probability W is an independent set of vertices that contains a point from each V_i. For each i, $1 \leqq i \leqq r$, let S_i be the event that $W \cap V_i = \emptyset$. Clearly $\Pr(S_i) = (1-p)^g$. For each edge f of H, let A_f be the event that W contains both ends of f. Clearly, $\Pr(A_f) = p^2$. Moroever, each event S_i is mutually independent of all the events

$$\{S_j : 1 \leqq j \leqq r, j \neq i\} \cup \{A_f : f \cap V_i = \emptyset\}.$$

Similarly, each event A_f is mutually independent of all the events

$$\{S_j : S_j \cap f = \emptyset\} \cup \{A_{f'} : f' \cap f = \emptyset\}.$$

Therefore, there is a dependency graph for the events $\{S_i : 1 \leqq i \leqq r\} \cup \{A_f : f \in E\}$ in which each S_j-node is adjacent to at most $g \cdot d$ $A_{f'}$-nodes (and to no S_j-nodes), and each A_f-node is adjacent to at most 2 S_j-nodes, and at most $2d-2$ $A_{f'}$-nodes. It follows from Lemma 2.3 that if we can find two numbers x and y, $0 \leqq x < 1$, $0 \leqq y < 1$ so that

$$(2.1) \qquad (1-p)^g = \left(1 - \frac{1}{25d}\right)^{25d} = \Pr(S_i) < x(1-y)^{gd} = x(1-y)^{25d^2}$$

and

$$(2.2) \qquad p^2 = \frac{1}{(25d)^2} = \Pr(A_f) < y(1-y)^{2d-2} \cdot (1-x)^2$$

then $\Pr(\wedge_{f\in E}\bar{A}_f \wedge_{1\leqq i\leqq r}\bar{S}_i)>0$. One can easily check that $x=\frac{1}{2}$, $y=1/100d^2$ satisfy (2.1) and (2.2). Indeed

$$\left(\frac{1}{2}\right)\left(1-\frac{1}{100d^2}\right)^{25d^2}\geqq\frac{1}{2}\left(1-\frac{25d^2}{100d^2}\right)=\frac{3}{8}\geqq\frac{1}{e}\geqq\left(1-\frac{1}{25d}\right)^{25d}$$

and

$$\frac{1}{100d^2}\left(1-\frac{1}{100d^2}\right)^{2d-2}\left(\frac{1}{2}\right)^2\geqq\frac{1}{400d^2}\left(1-\frac{1}{50d}\right)>\frac{1}{(25d)^2}.$$

Therefore,

$$\Pr\left(\bigwedge_{f\in E}\bar{A}_f\bigwedge_{1\leqq i\leqq r}\bar{S}_i\right)>0,$$

i.e., with positive probability, none of the events S_i or A_f hold for W. In particular, there is at least one choice for such $W\subseteq V$. But this means that this W is an independent set, containing at least one vertex from each V_i. This completes the proof. □

Proof of Theorem 2.1. Let $G=(U,F)$ be a d-regular graph with girth $g\geqq 50d$. By a well known theorem of Petersen ([Pe], see also [BM]), F can be partitioned into $d/2$ pairwise disjoint 2-factors $F_1,\ldots,F_{d/2}$. Each F_i is a union of cycles $C_{i1},C_{i2},\ldots,C_{ir_i}$. Let $V_1,V_2,\ldots,V_r$ be the sets of *edges* of all the cycles $\{C_{ij}:1\leqq i\leqq d/2,\ 1\leqq j\leqq r_i\}$. Clearly $V_1,V_2,\ldots,V_r$ is a partition of the set F of all edges of G, and by the girth condition, $|V_i|\geqq g\geqq 50d$ for all $1\leqq i\leqq r$. Let H be the line graph of G, i.e., the graph whose set of vertices is the set F of edges of G and two edges are adjacent iff they share a common vertex in G. Clearly H is $2d-2$ regular. As the cardinality of each V_i is at least $50d\geqq 25(2d-2)$, there is, by Proposition 2.4, an independent set of H containing a member from each V_i. But this means that there is a matching M in G, containing at least one edge from each cycle C_{ij} of the 2-factors $F_1,\ldots,F_{d/2}$. Therefore $M,F_1\setminus M,F_2\setminus M,\ldots,F_{d/2}\setminus M$ are $d/2+1$ linear forests in G (one of which is a matching) that cover all its edges. Hence

$$\mathrm{la}(G)\leqq\frac{d}{2}+1.$$

As G has $|U|\cdot d/2$ edges and each linear forest can have at most $|U|-1$ edges,

$$\mathrm{la}(G)\geqq|U|\frac{d}{2}\Big/(|U|-1)>\frac{d}{2}.$$

Thus $\mathrm{la}(G) = d/2 + 1$, completeing the proof. □

Two easy corollaries of Theorem 2.1, which will be useful in the next section, are the following.

COROLLARY 2.6. *Let G be a graph with maximum degree Δ and girth $g \geqq 100 \cdot \lceil \Delta/2 \rceil$. Then $\lceil \Delta/2 \rceil \leqq \mathrm{la}(G) \leqq \lceil \Delta/2 \rceil + 1$.*

PROOF. The lower bound is obvious, as any linear forest contains at most two edges incident with a vertex of maximum degree in G. To prove the upper bound, observe that it is always possible to add vertices and edges to G and get a $2\lceil \Delta/2 \rceil$-regular graph H with girth g. By Theorem 2.1, the linear arboricity of this new graph H is precisely $\lceil \Delta/2 \rceil + 1$. As G is a subgraph of H we conclude that $\mathrm{la}(G) \leqq \mathrm{la}(H) = \lceil \Delta/2 \rceil + 1$. □

COROLLARY 2.7. *Let $G = (V, E)$ be a graph with girth g and maximum degree $\Delta \geqq 2$, where $5000\Delta \geqq g^2$. Then*

$$\mathrm{la}(G) \leqq \frac{\Delta}{2} + \frac{200\Delta}{g}. \tag{2.3}$$

PROOF. By the well known theorem of Vizing ([V], see also [BM]) the edges of G can be partitioned into $\Delta + 1$ pairwise disjoint matchings $M_1, M_2, \ldots, M_{\Delta+1}$. This (as well as many other trivial arguments) suffices to show that $\mathrm{la}(G) \leqq \Delta + 1 \leqq 3\Delta/2$, which implies inequality (2.3) for every $g \leqq 200$. Hence we may asume that $g > 200$. Put $r = 2\lfloor g/100 \rfloor$ and split the set of the $\Delta + 1$ matchings $M_1, \ldots, M_{\Delta+1}$ into $s = \lceil (\Delta + 1)/r \rceil$ pairwise disjoint sets $S_1, \ldots, S_s$, each containing at most r matchings. For $1 \leqq i \leqq s$, let G_i be the subgraph of G consisting of all edges in $\bigcup_{j \in S_i} M_j$. The s graphs $G_1, \ldots, G_s$ cover all edges of G. Moreover, the maximum degree in each G_i is at most r, and its girth is at least $g \geqq 100\lceil r/2 \rceil$. Therefore, by Corollary 2.6, the linear arboricity of each G_i is at most $\lceil r/2 \rceil + 1 = \lfloor g/100 \rfloor + 1$. Consequently

$$\begin{aligned}
\mathrm{la}(G) &\leqq \sum_{i=1}^{s} \mathrm{la}(G_i) \leqq s \cdot (\lfloor g/100 \rfloor + 1) \leqq \left(\frac{\Delta+1}{2 \cdot \lfloor g/100 \rfloor} + 1\right)\left(\frac{g}{\lfloor 100 \rfloor} + 1\right) \\
&= \frac{\Delta}{2} + \frac{\Delta+1}{2\lfloor g/100 \rfloor} + \left\lfloor \frac{g}{100} \right\rfloor + \frac{3}{2} \leqq \frac{\Delta}{2} + \frac{100(\Delta+1)}{g} + \frac{g}{100} + \frac{3}{2} \\
&\leqq \frac{\Delta}{2} + \frac{100\Delta}{g} + \frac{g}{100} + 2 \leqq \frac{\Delta}{2} + \frac{200\Delta}{g},
\end{aligned}$$

where in the last three inequalities we used the fact that $g \geqq 200$ and $5000\Delta \geqq g^2$ imply that

$$\frac{1}{2\lfloor g/100 \rfloor} \leqq \frac{100}{g} \quad \text{and} \quad \frac{g}{100} + 2 \leqq \frac{100\Delta}{g}.$$

This completes the proof. □

We conclude this section with the following proposition, that shows that under certain conditions Conjecture 1.1 holds for an odd degree of regularity, too. The proof here is similar to that of Theorem 2.1, but is somewhat more complicated.

PROPOSITION 2.8. *Let $G = (U, F)$ be a d-regular graph, where $d = 2k + 1$ is an odd integer, and with girth $g \geqq 100d$. Suppose, further, that G contains a perfect matching F_0. Then*

$$\mathrm{la}(G) = k + 1 = \frac{d+1}{2}.$$

PROOF. By applying Petersen's Theorem to the $2k$-regular graph $(U, F - F_0)$ we conclude that there is partition of F into $k+1$ pairwise disjoint sets $F_0, F_1, \ldots, F_k$, where F_0 is the given matching and each F_i is a 2-factor of G. For i, $1 \leqq i \leqq k$, let $C_{i1}, C_{i2}, \ldots, C_{ir_i}$ be the cycles in F_i. Let $V_1, \ldots, V_r$ be the sets of edges of all the cycles $\{C_{ij} : 1 \leqq i \leqq k,\ 1 \leqq j \leqq r_i\}$. Recall that by the girth condition $|V_i| \geqq g \geqq 100d$ for all $1 \leqq i \leqq r$. We now construct a graph $H = (V(H), E(H))$ as follows. The vertex set $V(H)$ of H is $V_1 \cup \cdots \cup V_r = F - F_0$. Two vertices e, f of H (which are simply two edges of G that are not in the matching F_0) are adjacent in H iff there is an edge of F_0 which is adjacent (in G) with both of them. (In particular, if e and f share a common vertex in G they are adjacent in H.) One can easily check that H is $(4d-6)$-regular. As the cardinality of each V_i is a least $100d \geqq 25(4d-6)$, there is, by Proposition 2.4, an independent set W in H, containing a member from each V_i. But this means that W is a set of edges in $F - F_0$, that contains at least one edge from each cycle of each of the 2-factors $F_1, \ldots, F_k$, and contains no two edges incident with the same edge of F_0. Consequently, $F_0 \cup W$ is a linear forest (with connected components of length 1 or 3 each), and $F_1 \setminus W, F_2 \setminus W, \ldots, F_k \setminus W$ are also linear forests. Hence $\mathrm{la}(G) \leqq k + 1 = (d+1)/2$. This, together with the trivial inequality $\mathrm{la}(G) > d/2$ shows that $\mathrm{la}(G) = (d+1)/2$, completing the proof. □

3. The general case

The main result of this section is the following theorem.

THEOREM 3.1. *For $\varepsilon > 0$ there is a $d_0 = d_0(\varepsilon)$ such that for all $d > d_0$, the linear arboricity of any d-regular graph G satisfies*

$$\tfrac{1}{2}d < \mathrm{la}(G) < (\tfrac{1}{2} + \varepsilon)d.$$

Notice that this theorem implies that for every $\varepsilon > 0$ and every graph G with maximum degree $\Delta > \Delta_0(\varepsilon)$ the inequality $\mathrm{la}(G) < (\frac{1}{2} + \varepsilon)\Delta$ holds.

To prove the theorem, we need the following lemma, which shows that every regular graph contains an almost regular spanning subgraph with relatively large girth.

LEMMA 3.2. *For all sufficiently large d, any d-regular graph $G = (V, E)$ contains a spanning subgraph $H = (V, F)$ with the following two properties:*

(i) *The girth g of H satisfies*

$$g \geqq \log d / 20 \log\log d. \tag{3.1}$$

(Here and throughout the paper all logarithms are in base e.)

(ii) *For every vertex $v \in V$, the degree $d_H(v)$ of v in H satisfies*

$$\lceil \log^{10} d - \log^6 d \rceil \leqq d_H(v) \leqq \lfloor \log^{10} d + \log^6 d \rfloor. \tag{3.2}$$

PROOF. In the proof we assume, whenever it is needed, that d is sufficiently large. Define

$$s = \log d / 20 \log\log d \qquad \text{and} \qquad p = d^{1/2s-1} = \frac{\log^{10} d}{d}.$$

Clearly $0 < p < 1$. Let us pick each edge of G, randomly and independently, with probability p, to get a random set F of all the edges picked. To complete the proof we show, using, again, the Lovász Local Lemma (Lemma 2.3), that with positive probability $H = (V, F)$ satisfies (3.1) and (3.2). For every cycle C of length at most s in G, let A_C be the event that F contains C. Similarly, for every vertex $v \in V$, let B_v be the event that

$$| d_H(v) - \log^{10} d | > \log^6 d.$$

Clearly, for every cycle C of length k, where $3 \leqq k \leqq s$

$$(3.3)\qquad \Pr(A_C) = p^k = d^{k/2s-k} \leqq \frac{1}{d^{k-1/2}}.$$

Similarly, by the standard estimates for Binomial distributions (see, e.g., [B]), for every $v \in V$

$$(3.4)\qquad \Pr(B_v) \leqq e^{-\log^2 d/2} = \frac{1}{d^{s\,10\log\log d}} < \frac{1}{d^{10s}}.$$

Let $\mathscr{C}$ denote the set of all cycles of length at most s in G. Define a dependency graph T on the set of vertices $\{A_C : C \in \mathscr{C}\} \cup \{B_v : v \in V\}$ as follows. A_C and $A_{C'}$ are adjacent iff the two cycles C and C' share a common edge in G. B_v and $B_{v'}$ are adjacent iff v and v' are adjacent in G. A_C and B_v are adjacent iff v is a vertex in the cycle C. One can easily check that T is a dependency graph for $\{A_C : C \in \mathscr{C}\} \cup \{B_v : v \in V\}$. For each k, $3 \leqq k \leqq s$, let $\mathscr{C}_k$ be the set of all cycles of length k in G. Clearly $\mathscr{C} = \bigcup\{\mathscr{C}_k : 3 \leqq k \leqq s\}$. Notice that since G is d-regular, the number of cycles of length r that contain a given vertex of G is at most d^{r-1}, whereas the number of cycles of length r that contain a given edge of G is at most d^{r-2}. Consequently, every B_v-node in T is adjacent in T to at most d^{r-1} A_C-nodes with $C \in \mathscr{C}_r$. Also, every B_v-node is adjacent in T to precisely d other $B_{v'}$-nodes. Similarly, if $C \in \mathscr{C}_k$, every A_C-node is adjacent in T to at most k B_v-nodes, and to at most kd^{r-2} $A_{C'}$-nodes corresponding to cycles $C' \in \mathscr{C}_r$. We next apply Lemma 2.3 with the real numbers $0 \leqq x_C < 1$ and $0 \leqq y_v < 1$ defined as follows. For each $v \in V$, $y_v = 1/d^s$. For each $C \in \mathscr{C}_k$, $x_C = 1/d^{k-1}$. In view of the last paragraph, inequalities (3.3) and (3.4) and Lemma 2.3, the inequality

$$\Pr\Big(\bigwedge_{c\in C} \bar{A}_c \bigwedge_{v\in V} \bar{B}_v\Big) > 0$$

holds, provided the following inequalities (3.5) and (3.6) hold:

$$(3.5)\qquad \frac{1}{d^{k-1/2}} < \frac{1}{d^{k-1}} \prod_{r=3}^{s} \left(1 - \frac{1}{d^{r-1}}\right)^{kd^{r-2}} \cdot \left(1 - \frac{1}{d^s}\right)^k \qquad (3 \leqq k \leqq s),$$

$$(3.6)\qquad \frac{1}{d^{10s}} < \frac{1}{d^s} \prod_{r=3}^{s} \left(1 - \frac{1}{d^{r-1}}\right)^{d^{r-1}} \cdot \left(1 - \frac{1}{d^s}\right)^d.$$

Recall that d is large and that $s = \log d/20 \log\log d$. Therefore, for each fixed k, $3 \leqq k \leqq s$,

$$\frac{1}{d^{k-1}}\prod_{r=3}^{s}\left(1-\frac{1}{d^{r-1}}\right)^{kd^{r-2}}\cdot\left(1-\frac{1}{d^s}\right)^k \geqq \frac{1}{d^{k-1}}\left(1-\frac{k}{d^s}\right)\prod_{r=3}^{s}\left(1-\frac{k}{d}\right)$$

$$> \frac{1}{d^{k-1}}\left(1-\frac{s}{d}\right)\cdot\left(1-\frac{s^2}{d}\right)$$

$$> \frac{1}{d^k}$$

and inequality (3.5) holds. Similarly

$$\frac{1}{d^s}\prod_{r=3}^{s}\left(1-\frac{1}{d^{r-1}}\right)^{d^{r-1}}\cdot\left(1-\frac{1}{d^s}\right)^d \geqq \frac{1}{d^s}\frac{1}{4^s}\cdot\frac{1}{2} > \frac{1}{d^{10s}},$$

establishing (3.6). We conclude that with positive probability none of the events A_C or B_v hold for $H=(V,F)$. In particular, there is at least one choice for such an H. But this means that H is a spanning subgraph of G that satisfies (3.1) and (3.2). This completes the proof. □

We can now prove the following Proposition, which clearly implies Theorem 3.1.

Proposition 3.3. *There exists a constant $c>0$, such that the linear arboricity of any d-regular graph $G=(V,E)$ satisfies*

$$\tfrac{1}{2}d < \mathrm{la}(G) < \tfrac{1}{2}d + \frac{6000d\cdot\log\log d}{\log d} + c.$$

Proof. The lower bound is trivial, since G has $|V|d/2$ edges, and each linear forest in G contains at most $|V|-1$ edges.

To prove the upper bound we argue as follows. Let c_1 be a constant so that the assertion of Lemma 3.2 holds for every $d \geqq c_1$. Let c_2 be a constant so that for every $d \geqq c_2$ the following inequality holds: Put $\bar{d} = d - \lceil \log^{10} d - \log^6 d \rceil$, then

$$6000\left(\frac{d\log\log d}{\log d} - \frac{\bar{d}\log\log \bar{d}}{\log \bar{d}}\right) \geqq \log^6 d + \frac{4000(\log^{10} d + \log^6 d)\log\log d}{\log d}, \tag{3.7}$$

Note that it is not too difficult to check that such c_2 exists. This is because if

$$f(x) = \frac{x\log\log x}{\log x}$$

then, as x tends to infinity,

$$f'(x)=\frac{\log\log x}{\log x}+x\frac{(\log x)\cdot\frac{1}{\log x}\cdot\frac{1}{x}-(\log\log x)\cdot\frac{1}{x}}{\log^2 x}$$

$$=\frac{\log\log x}{\log x}(1+o(1)).$$

Therefore, by the mean-value theorem, for large d there is some d', $\bar{d}\leqq d'\leqq d$ so that the left-hand side of (3.7) is

$$6000(1+o(1))\cdot(d-\bar{d})\cdot\frac{\log\log d'}{\log d'}$$

$$\geqq(1+o(1))6000(\log^{10} d-\log^6 d)\frac{\log\log d}{\log d}.$$

The last quantity is clearly bigger, for sufficiently large d, than the right-hand side of (3.7). Therefore there is a $c_2>0$ so that for $d\geqq c_2$, (3.7) holds. We now prove the upper bound in Proposition 3.3 with $c=\max(100, c_1, c_2)$ by induction on d. For $d\leqq c$ the inequality is trivial. Thus we may assume the upper bound for all $d'<d$, and prove it for d, where $d\geqq c$. Let $G=(V,E)$ be a d-regular graph. Since $d\geqq c_1$ we may apply Lemma 3.2 to conclude that there is a spanning subgraph $H=(V,F)$ of G satisfying (3.1) and (3.2). H clearly satisfies the assumptions of Corollary 2.7, and hence, by that Corollary and by the bounds (3.1), (3.2) for the girth of H and its maximum degree:

$$\text{(3.8)}\qquad \mathrm{la}(H)\leqq\frac{\log^{10} d+\log^6 d}{2}+\frac{4000(\log^{10} d+\log^6 d)\log\log d}{\log d}.$$

Let $T=(V,E-F)$ be the graph obtained from G by deleting from it the edges of H. By (3.2), the maximum degree in T is at most $\bar{d}=d-\lceil\log^{10} d-\log^6 d\rceil$. Therefore one can add, if necessary, edges and vertices to T to embed it in a $\bar{d}$ regular graph. By applying the induction hypothesis we get an upper bound for the linear arboricity of this new graph, which is clearly also an upper bound for the linear arboricity of T. This gives

$$\text{(3.9)}\qquad \mathrm{la}(T)\leqq\tfrac{1}{2}\bar{d}+\frac{6000\bar{d}\log\log\bar{d}}{\log\bar{d}}+c.$$

Combining (3.8) and (3.9) we obtain

$$\begin{aligned}
\mathrm{la}(G) &\leqq \mathrm{la}(H) + \mathrm{la}(T) \\
&\leqq \tfrac{1}{2}d + \frac{6000d \log\log d}{\log d} + c + \frac{\log^{10} d + \log^6 d}{2} \\
&\quad + \frac{4000(\log^{10} d + \log^6 d)\cdot \log\log d}{\log d} \\
&\leqq \tfrac{1}{2}d + \frac{6000d \log\log d}{\log d} + c,
\end{aligned}$$

where the last inequality follows from inequality (3.7), which holds since $d \geqq c_2$. This completes the proof of the induction step, and the assertion of Proposition 3.3 (as well as that of Theorem 3.1) follows. □

4. Related results

(1) A *d-regular digraph* is a directed graph in which the indegree and the outdegree of every vertex is precisely d. A linear directed forest is a directed graph in which every connected component is a directed path. The *di-linear arboricity* $\mathrm{dla}(G)$ of a directed graph G is the minimum number of linear directed forests in G whose union covers all edges of G. In [NP] the authors conjecture that for every d-regular digraph G, $\mathrm{dla}(G) = d + 1$, and prove this conjecture for $d \leqq 2$. This easily implies that for every d-regular digraph G,

$$d + 1 \leqq \mathrm{dla}(G) \leqq 3\lceil d/2 \rceil.$$

The proofs of Theorems 2.1 and 3.1 can be easily modified to establish the following two propositions, whose detailed proof is omitted.

PROPOSITION 4.1. *Let G be a d-regular graph with no directed cycles of length smaller than $50d$. Then $\mathrm{dla}(G) = d + 1$. Moreover, the edges of G can be covered by d linear directed forests and a matching.* □

PROPOSITION 4.2. *For every $\varepsilon > 0$ there is a $d_0 = d_0(\varepsilon)$ such that for every $d > d_0$ and every d-regular digraph G the inequality*

$$d + 1 \leqq \mathrm{dla}(G) \leqq (1 + \varepsilon)d$$

holds. □

(2) In [AD], [AS] the authors consider the linear arboricity $\mathrm{la}(G)$ of a

loopless multigraph G, that is, the minimum number of linear forests in G whose union covers all edges of G. The analogue of Conjecture 1.1 here is that for every loopless multigraph G with maximum degree Δ and maximum edge-multiplicity μ we have

$$\mathrm{la}(G) \leqq \left\lceil \frac{\Delta + \mu}{2} \right\rceil .$$

By a simple modification of Theorem 3.1 one can prove the following result, whose detailed proof is omitted.

PROPOSITION 4.3. *For every fixed μ and ε there is a $\Delta_0 = \Delta_0(\mu, \varepsilon)$ so that for every $\Delta > \Delta_0$, the linear arboricity of every loopless multigraph G with maximum degree Δ and maximum edge multiplicity μ is at most $(\frac{1}{2} + \varepsilon)\Delta$.* □

(3) A *k-linear forest* is a forest whose connected components are paths of length k or less. The *k-linear arboricity* $\mathrm{la}_k(G)$ of a (simple, undirected) graph G is the minimum number of k-linear forests whose union is the set of all edges of G. This notion is introduced in [HP1] and studied in [HP2], [BFHP]. The analogue of Conjecture 1.1 for this case is raised in [HP1]. Applying our method we can prove here that for every graph G, with an even degree of regularity d, with girth $g \geqq 50d$ and for every $k \geqq 100d$ $\mathrm{la}_k(G) = \mathrm{la}(G) = d/2 + 1$. A somewhat complicated analogue of Theorem 3.1 for the function $\mathrm{la}_k(G)$ can also be formulated and proved.

(4) A *star forest* is a forest whose connected components are stars. The *star arboricity* $\mathrm{st}(G)$ of a graph G is the minimum number of star forests whose union is the set of all edges of G. This notion is introduced in [AK], where it is shown that the star arboricity of the complete graph on n vertices is $\lceil n/2 \rceil + 1$. In [Ao] it is shown that for every complete-multipartite graph G with equal color classes, the star arboricity does not exceed $\lceil d/2 \rceil + 2$, where d is the degree of regularity of G. Notice that trivially for every d-regular graph G, $\mathrm{st}(G) > d/2$. In view of the two results stated above, and in analogy to the linear arboricity conjecture, one may be tempted to conjecture that for every d-regular graph G, $d/2 < \mathrm{st}(G) < d/2 + c$ for some constant c. However, as we show in a forthcoming paper [AA] this is not the case. There are d-regular graphs G for which $\mathrm{st}(G) > d/2 + \Omega(\log d)$. On the other hand, by applying probabilistic methods in a similar way to the one done in this paper, we show in [AA] that for every $\varepsilon > 0$ the star arboricity of any d-regular graph G does not exceed $(\frac{1}{2} + \varepsilon)d$, provided $d > d_0(\varepsilon)$.

Acknowledgement

I would like to thank Y. Roditty for telling me about many of the papers dealing with linear arboricity. I am also grateful to Y. Caro for helpful comments.

Note added in proof. By replacing Lemma 3.2 by another application of the Local Lemma we can improve the estimate in Proposition 3.3 to $\mathrm{la}(G) \leqq \frac{1}{2}d + O((d \log d)^{2/3})$. Another recent result, related to Proposition 2.4, is: Let $H = (V, E_1 \cup E_2)$ be a graph, where E_1 is a union of d matchings, and E_2 is a union of vertex disjoint cliques, of size 2^d each. Then the chromatic number of H is 2^d.

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